Student Number____________________

ASCHAM SCHOOL

2022

YEAR 12

TRIAL

EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black non-erasable pen.
- NESA-approved calculators may be used.
- A NESA Reference Sheet is provided.
- All necessary working should be shown in every question.

Total marks – 100

- Attempt Sections A and B.
- Section A is worth 10 marks.
- Recommended time on Section A: 15 minutes
- Answer Section A on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section B contains 6 questions worth 15 marks each.
- Recommended time on Section B: 2 hours 45 minutes
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

SECTION A – 10 MULTIPLE CHOICE QUESTIONS 10 MARKS

ANSWER ON THE ANSWER SHEET

1 Which of the following vectors is perpendicular to the vector $\underset{\sim}{u}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and the *y*-axis?

A $\underset{\sim}{m}=\begin{pmatrix}3\\0\\-1\end{pmatrix}$

B $\underset{\sim}{m}=\begin{pmatrix}0\\3\\-2\end{pmatrix}$

C $\underset{\sim}{m}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$

D $\underset{\sim}{m}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$

2 What is the minimum value of $|2x-y|+|x+2y|$?

A $|x+3y|$

B $|x-3y|$

C $|3x-y|$

D $|3x+y|$

3 Which of the following is true?

A $\forall x, y \in \mathbb{N}, \exists a, b \in \mathbb{Q} : \frac{1}{x+iy} = a+ib$

B $\forall x, y \in \mathbb{R}, \exists a, b \in \mathbb{Z} : \frac{1}{x+iy} = a+ib$

C $\forall x, y \in \mathbb{Q}, \exists a, b \in \mathbb{N} : \frac{1}{x+iy} = a+ib$

D $\forall x, y \in \mathbb{Z}, \exists a, b \in \mathbb{Z} : \frac{1}{x+iy} = a+ib$

4 It is known that $1-2i$ is a root of $z^4 + az^3 + bz^2 + 10 = 0$, where a and b are real. Which of the following could also be a root of $z^4 + az^3 + bz^2 + 10 = 0$?

A $1+i$

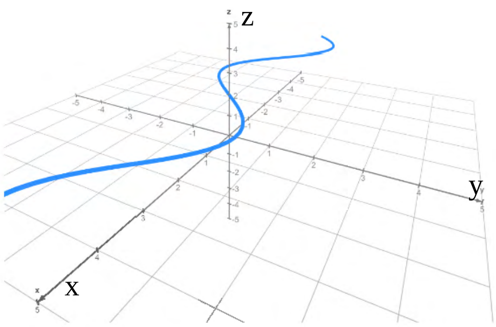
B 2

C $1+2i$

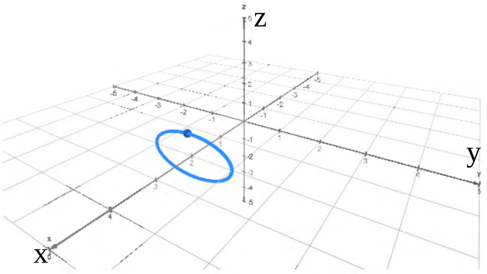
D All of the above

5 What is a possible graph for the parametric equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ t \\ \cos t \end{pmatrix}$?

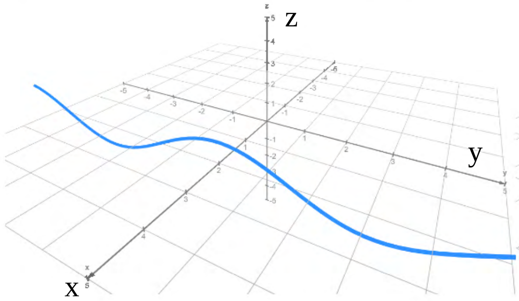
A



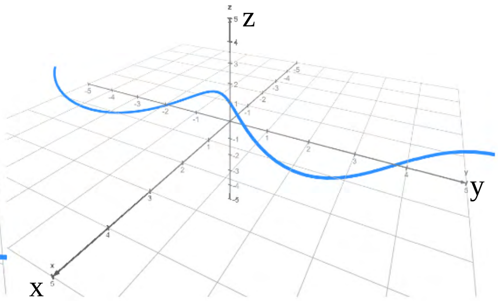
B



C



D



6 Which of the following is the contrapositive of the following statement?
$\forall x, y, z, n \in \mathbb{N}$: *if* $x^n + y^n \neq z^n$ *then* $n \geq 3$.

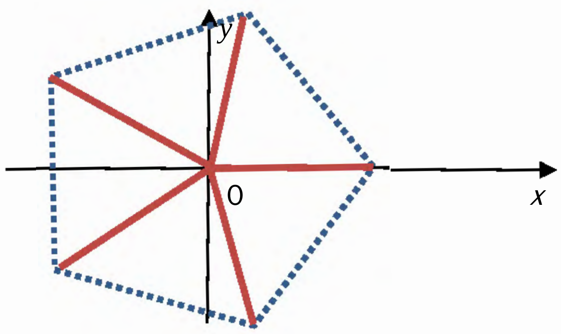
A $\forall x, y, z, n \in \mathbb{N}$: *if* $n \geq 3$ *then* $x^n + y^n \neq z^n$.

B $\forall x, y, z, n \in \mathbb{N}$: *if* $n < 3$ *then* $x^n + y^n \neq z^n$.

C $\forall x, y, z, n \in \mathbb{N}$: *if* $n \geq 3$ *then* $x^n + y^n = z^n$.

D $\forall x, y, z, n \in \mathbb{N}$: *if* $n < 3$ *then* $x^n + y^n = z^n$.

7 The graph below shows roots of unity on the Argand diagram.



What is the equation?

A $z^4 = -1$

B $z^5 = -1$

C $z^4 = 1$

D $z^5 = 1$

8 What is the value of $\sum_{1}^{99} i^n$?

A i

B -1

C $-i$

D 1

9 Consider the complex numbers z such that $|z-2i|=1$. What is the maximum value of $\arg z$?

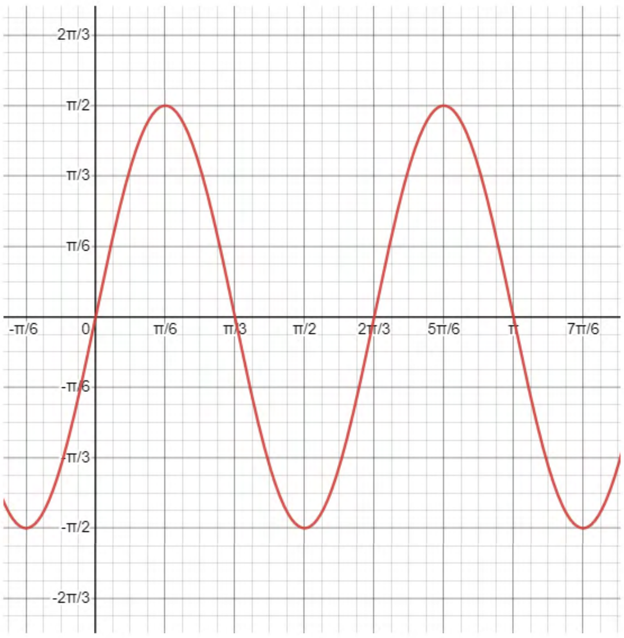
A $\dfrac{\pi}{4}$

B $\dfrac{\pi}{3}$

C $\dfrac{2\pi}{3}$

D $\dfrac{3\pi}{4}$

10 A particle P is moving in simple harmonic motion. The graph of **velocity** versus time is given below.



What is the amplitude of the motion?

A $\dfrac{\pi}{12}$

B $\dfrac{\pi}{6}$

C $\dfrac{\pi}{3}$

D $\dfrac{\pi}{2}$

SECTION 2 – 6 QUESTIONS EACH WORTH 15 MARKS

Question 11 – Begin a new writing booklet

a Find $\int x\sqrt{1-x}\ dx$. **3**

b Find $\int_1^2 \frac{4}{x\left(x^2+4\right)}\,dx$. **3**

c Find $\int_0^1 \frac{dw}{1+\sqrt{w}}$. **3**

d

The 4 complex numbers z_1, z_2, z_3, z_4 are represented by the points Z_1, Z_2, Z_3, Z_4 on an Argand diagram. $Z_1Z_2Z_3Z_4$ is a square.

i Prove that $\left|z_3 - z_1\right| - \left|z_4 - z_2\right| = 0$. **2**

ii Prove that $\arg\left(\frac{z_4 - z_2}{z_3 - z_1}\right) = \frac{\pi}{2}$. **2**

iii Prove that $z_3 - z_1 = (1+i)(z_2 - z_1)$. **2**

Question 12 – Begin a new writing booklet

a Consider the statement:

$\forall A \in \mathbb{R}$: If $\sin A = k$ then $\sin^{-1} k = A$.

i Write the converse. **2**

ii Write the contrapositive. **2**

iii Write the negation. **2**

iv Determine whether or not the statement is an equivalence. Give reasons. **2**

v Determine whether or not this statement is true. If true, justify using logic, or if not, give a counter-example. **2**

c For the complex number $z = x + iy$ where $x, y \in \mathbb{R}$ sketch the set of z such that:

i $|z+1+i| = |z-1-i|$ **2**

ii $|z+1+i| \leq 1$ **1**

iii $z^4 = 16e^{i\pi}$. **2**

Question 13 – Begin a new writing booklet

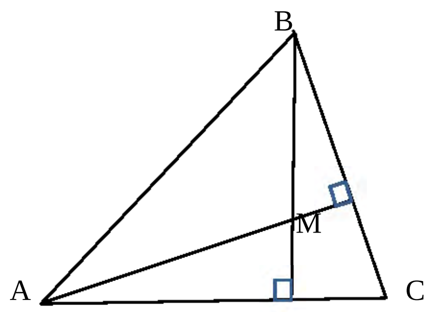
a The two lines $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 6 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ intersect at (a,b,c).

i Find (a,b,c). **3**

ii Show that $\underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular. **2**

iii Find a vector which is perpendicular to both $\underset{\sim}{r}$ and $\underset{\sim}{q}$. **3**

b Consider the triangle *ABC* shown. The altitudes (perpendicular heights) from *A* to *BC* and *B* to *AC* intersect at *M*, i.e. $(\underset{\sim}{a} - \underset{\sim}{m}) \perp (\underset{\sim}{b} - \underset{\sim}{c})$, etc. **4**



Copy the diagram.

Use vectors to prove that *CM* is perpendicular to *AB*.

[This is the proof of the theorem that the altitudes of a triangle are concurrent.]

c Let $z = \cos\theta + i\sin\theta$. Using any method, factorise $z^6 + 1$ into 3 quadratic factors with real coefficients. **3**

Question 14 – Begin a new writing booklet

a **i** If a, b are real, show that $a^2+b^2 \geq 2ab$. **1**

Hence, or otherwise, show that if a, b, c are real then

ii $a^2+b^2+c^2 \geq ab+bc+ca$ **2**

iii $\left(a^2+b^2+c^2\right)^2 \geq 3abc\left(a+b+c\right)$. **2**

b Let $I_n=\int_0^1 x\left(1-x^3\right)^n dx$.

i Show that $I_n=\dfrac{3n}{3n+2}I_{n-1}$. **3**

ii Use the identity above to find I_3 . **2**

c **i** Show that $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ for $|x|<1$ and $|y|<1$. **1**

ii Use mathematical induction to prove that **3**

$$\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right)=\tan^{-1}\left(\frac{n}{n+1}\right).$$

iii Find $\lim\limits_{n\to\infty}\sum_{j=1}^{n}\tan^{-1}\left(\dfrac{1}{2j^2}\right)$. **1**

Question 15 – Begin a new writing booklet

a A suitcase of mass m kg on wheels is in equilibrium sitting on a flat rough surface. A person is trying to pull the suitcase at an angle of 60° to the horizontal with force S Newton. Gravity of g m/s^2 is acting downwards.

i Draw a diagram and resolve the forces in the horizontal and vertical directions. **2**

ii Find an expression for the coefficient μ of static frictional force F Newton on the object in terms of S, m and g. **2**

b A particle is moving horizontally at time t seconds, such that $v^2 = x(6-x)$ where v is velocity in m/s and x is displacement in metres.

i Prove that it is moving in simple harmonic motion. **2**

ii Find the centre and amplitude. **2**

iii Find the period. **1**

iv Find the maximum speed. **1**

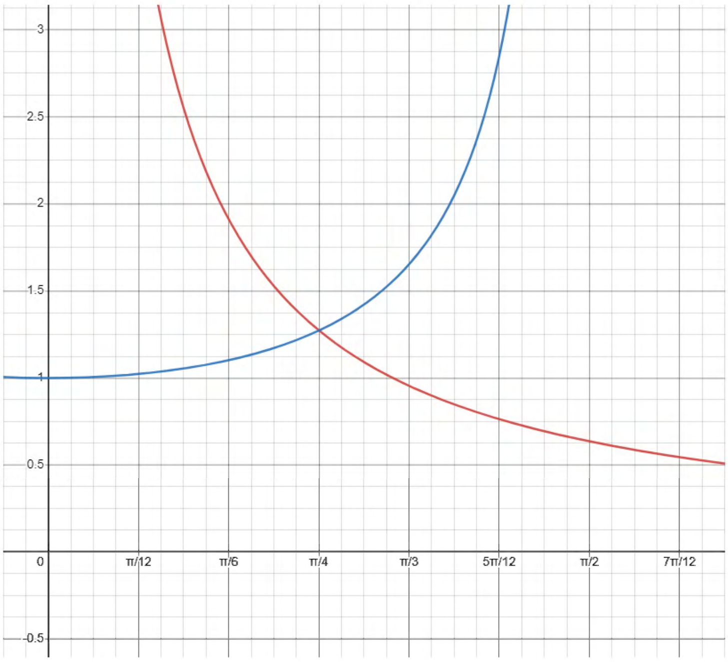
v At $t = 0, v = 3$ and $x = 3$. Find a formula for displacement x in terms of t. **2**

vi After 3 seconds, determine where the particle is, if the particle is travelling to the left or right, and if it is speeding up or slowing down. **3**

Question 16 – Begin a new writing booklet

a Use the substitution $u = \cos 2\theta$ to evaluate $\int_{\frac{1}{2}}^{1} \sqrt{\frac{1-u}{1+u}}\, du$. **4**

b Consider the graphs with equations $y = \frac{\tan x}{x}$ and $y = \frac{1}{x}$ for $x, y > 0$ shown below.



i Show that the curves intersect at $\left(\frac{\pi}{4}, \frac{4}{\pi}\right)$. **1**

ii Show with justification that $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}\, dx > \ln\left(\frac{4}{3}\right)$. **3**

Question 16 continues on the next page…

Question 16 continued…

c A particle with acceleration $\ddot{x}$, displacement x and velocity v at time t moves such that $\ddot{x} = \dfrac{v}{50}(50 - v)$. Initially the particle is at the origin and the velocity is 100.

i Find an equation for v in terms of x. **3**

ii Find an equation for x in terms of t. **2**

iii Find the time taken for the particle to travel 25 units. **2**

The end! ☺

Solutions to
Title: Ascham 2022 Y12 Trial Ext 2 Maths

MC:

1. y-axis is $\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\therefore \begin{pmatrix}3\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = 3\times0+0\times1-1\times0 = 0$

Test $\begin{pmatrix}3\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = 3\times1+0\times2+-1\times3 = 0$

$\therefore$ A

2. $|u|+|v| \geq |u+v|$ [Δ-ineq]

$\therefore |2x-y|+|x+2y| \geq |2x-y+x+2y|$
$\geq |3x+y|$

D

3. $\frac{1}{x+iy}\times\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = a+ib$

So $a = \frac{x}{x^2+y^2}$, $b = \frac{-y}{x^2+y^2}$

$\therefore a,b \in \mathbb{Q}$ if $x,y \in \mathbb{N}$

$\therefore$ A

4. $1-2i$ is root so $1+2i$ is.
$(1-2i)(1+2i) = 5$. Must be factors of 10 so other roots must be factors of 2.
$(1+i)(1-i) = 2$ or 2 $\therefore$ D

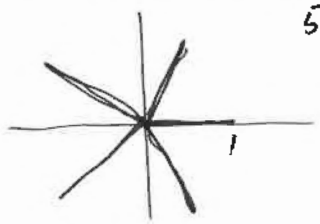
5. $x = 2$ $y = t$, $z = \cos t$

$\therefore z = \cos y$ and x fixed

$\therefore$ C

6. Contrapositive of $A \Rightarrow B$ is
$\neg B \Rightarrow \neg A$.
$\therefore$ If $n<3$ then $x^n+y^n=z^n$.

D

7. 

5 roots

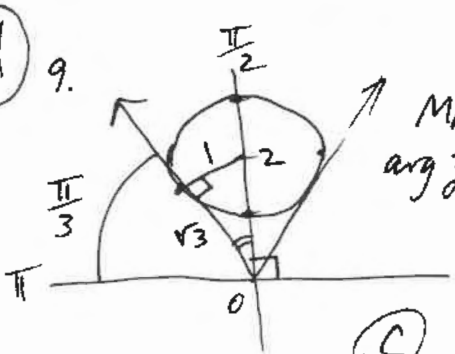
$\therefore z^5 = 1$

D

8. $\sum_1^{99} i^n = i^1+i^2+i^3+i^4+i^5+i^6+\ldots$
$\ldots + i^{96}+i^{97}+i^{98}+i^{99}$

$= \cancel{i}\cancel{-1}\cancel{-i}\cancel{+1}+i-1-i+1+i-\ldots$
$\ldots+\cancel{i}\cancel{-1}\cancel{-i}\cancel{+1}+\cancel{i}-1-i$

B

9. 

Max $\arg z = \frac{\pi}{2}+\frac{\pi}{6}$

$= \frac{2\pi}{3}$

C

10. Period $= \frac{2\pi}{3} = \frac{2\pi}{n} \Rightarrow n=3$.

Max vel $= \frac{\pi}{2}$.

let $x = -A\cos(nt+\varepsilon)$
$v = An\sin(nt+\varepsilon)$

$\therefore$ Max when $\sin(nt+\varepsilon)=1$

$\therefore v = An$
$\frac{\pi}{2} = A\times3$
$\therefore A = \frac{\pi}{6}$

B

SECTION 2

Q 11

a) $\int x\sqrt{1-x}\,dx$ let $u=1-x$, $du=-dx$

$=\int (1-u)\sqrt{u}\,.\,-du$

$=-\int u^{\frac{1}{2}}-u^{\frac{3}{2}}\,du$ (3)

$=-\left[\frac{2}{3}u^{\frac{3}{2}}-\frac{2}{5}u^{\frac{5}{2}}\right]+C$

$=\frac{2}{5}\sqrt{(1-x)^5}-\frac{2}{3}\sqrt{(1-x)^3}+C$

$=(1-x)\sqrt{1-x}\left[\frac{2}{5}(1-x)-\frac{2}{3}\right]+C$

$=(1-x)\sqrt{1-x}\left[-\frac{2}{5}x-\frac{4}{15}\right]+C$

$=(x-1)\sqrt{1-x}\left(\frac{2}{5}x+\frac{4}{15}\right)+C$

b) $\int_1^2 \frac{4}{x(x^2+4)}\,dx \equiv \int_1^2 \frac{A}{x}+\frac{Bx+C}{x^2+4}\,dx$

$\therefore A(x^2+4)+x(Bx+C)\equiv 4$

$B+A=0$, $4A=4$, $C=0$

$\therefore B=-1$ $A=1$

$\therefore \quad =\int_1^2 \frac{1}{x}+\frac{-x}{x^2+4}\,dx$

(3) $=\left[\ln|x|-\frac{1}{2}\ln|x^2+4|\right]_1^2$

$=\ln 2-\frac{1}{2}\ln 8-\ln 1+\frac{1}{2}\ln 5$

$=\ln 2-\frac{1}{2}\ln 8+\frac{1}{2}\ln 5$

$=\ln\left|\frac{2}{2\sqrt{2}\times\sqrt{5}}\right|$

$=-\frac{1}{2}\ln 10$

c) $\int_0^1 \frac{dw}{1+\sqrt{w}}$ let $u^2=w$, $2u\,du=dw$

$w=1$ $u=1$

$w=0$ $u=0$

$=\int_0^1 \frac{2u\,du}{1+u}$

Q11 c) Cont'd

$=2\int_0^1 \frac{u+1-1}{1+u}\,du$ (3)

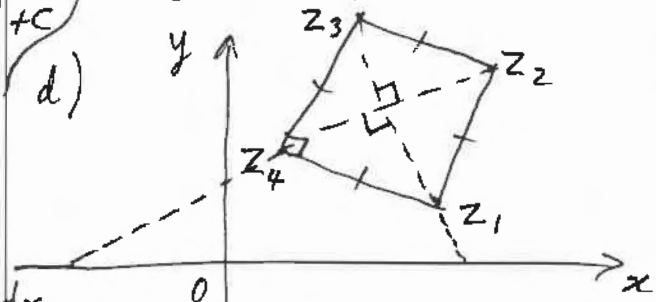
$=2\int_0^1 1-\frac{1}{1+u}\,du$

$=2\left[u-\ln|1+u|\right]_0^1$

$=2\left[1-\ln 2-0+\ln 1\right]$

$=2-2\ln 2$

d)



i) RTP: $|z_3-z_1|-|z_4-z_2|=0$

Proof: $Z_1Z_3=Z_4Z_2$ since it is a square (diagonals equal) (2)

$\therefore |z_3-z_1|=|z_4-z_2|$

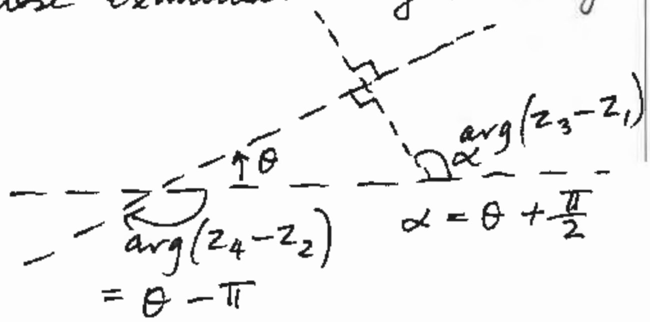
$\therefore |z_3-z_1|-|z_4-z_2|=0$.

ii) RTP: $\arg\left(\frac{z_4-z_2}{z_3-z_1}\right)=\frac{\pi}{2}$. (2)

Proof: $\arg\left(\frac{z_4-z_2}{z_3-z_1}\right)$

$=\arg(z_4-z_2)-\arg(z_3-z_1)$

Close examination of the diagram:



P.T.O.

Q 11 cont'd ii) Cont'd

$\therefore \arg(z_3 - z_1) = \alpha$

$\arg(z_4 - z_2) = \theta - \pi$ (negative)

$\therefore \arg(z_4 - z_2) - \arg(z_3 - z_1)$

$= \theta - \pi - \alpha$ but $\theta + \frac{\pi}{2} = \alpha$

$\therefore = \theta - \pi - (\theta + \frac{\pi}{2})$

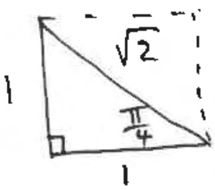
$= \cancel{\theta} - \pi - \cancel{\theta} - \frac{\pi}{2}$

$= -\frac{3\pi}{2} = \frac{\pi}{2}$

[Easier to say angle between 2 diagonals is 90° since square so $z_4 - z_2$ is $z_3 - z_1$ rotated 90° anticlockwise.]

iii) RTP: $z_3 - z_1 = (1+i)(z_2 - z_1)$

Proof: Multiply $z_2 - z_1$ by $1+i$ means rotating by $\frac{\pi}{4}$ $(\arg(1+i) = \frac{\pi}{4})$ and increasing by factor of $|1+i| = \sqrt{2}$.



Diagonal $z_3 - z_1$ is $|z_2 - z_1| \times \sqrt{2}$ and rotated 45°, anticlockwise. (2)

Q 12

a) $\forall A \in \mathbb{R}$: If $\sin A = k$ then $\sin^{-1} k = A$.

i) Converse: If $\sin^{-1} k = A$ then $\sin A = k$. (2)

ii) Contrapositive: If $\sin^{-1} k \neq A$ then $\sin A \neq k$. (2)

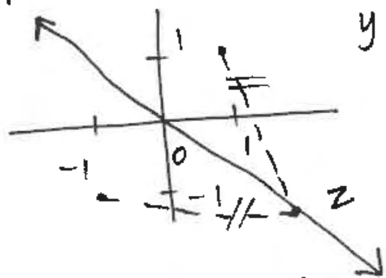
iii) Negation: Not all A such that $\sin A = k$ means $\sin^{-1} k = A$. or $\exists \sin A = k$ such that $\sin^{-1} k \neq A$. (2)

iv) Equivalence if $A \Rightarrow B$ and $B \Rightarrow A$ are both true. $A \Rightarrow B$ is NOT true ($\sin \frac{5\pi}{6} = \frac{1}{2}$ but $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$) but $B \Rightarrow A$ is true $\therefore$ Not an equivalence. (2)

v) Not true since contrapositive is not true. eg. $\sin^{-1} \frac{1}{\sqrt{2}} \neq \frac{3\pi}{4}$ but $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$. FALSE statement in the first place. (2)

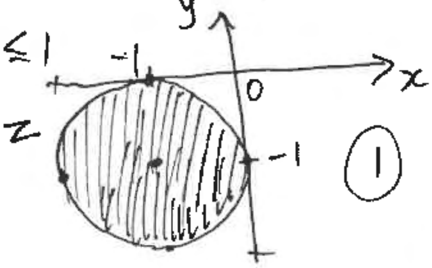
c) i) $|z+1+i| = |z-1-i|$

$y = -x$



(2)

ii) $|z+1+i| \leq 1$



(1)

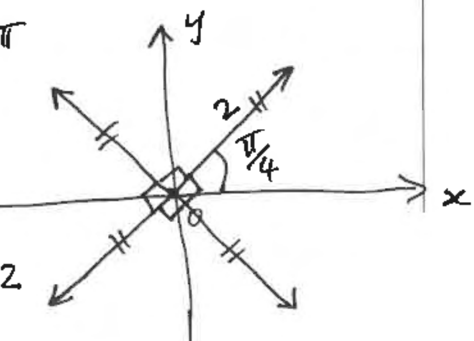
iii) $z^4 = 16e^{i\pi}$

$z^4 = -16$

$r^4 \operatorname{cis} 4\theta = -16$

(2) $\therefore r = 2$

$\theta = \frac{\pi}{4}$



Q13 a) $\underset{\sim}{r} = \begin{pmatrix}1\\-3\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}4\\-5\\2\end{pmatrix}$.

$\underset{\sim}{q} = \begin{pmatrix}-1\\6\\7\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\2\\3\end{pmatrix}$

$\therefore$ i) $1+4\lambda_1 = -1+\lambda_2 \Rightarrow \lambda_2 = 2+4\lambda_1$

$-3-5\lambda_1 = 6+2\lambda_2$ ②

$3+2\lambda_1 = 7+3\lambda_2$ ③

Sub $\lambda_2 = 2+4\lambda_1$ into ②

$-3-5\lambda_1 = 6+2(2+4\lambda_1)$

$-3-5\lambda_1 = 6+4+8\lambda_1$

$-13 = 13\lambda_1 \Rightarrow \lambda_1 = -1$

$\therefore \lambda_2 = 2+4(-1) = -2$

$\therefore \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-3\\2\\1\end{pmatrix}$ ③

ii) $\underset{\sim}{r}.\underset{\sim}{q} = \begin{pmatrix}4\\-5\\2\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}$ ②

$= 4\times1 - 5\times2 + 2\times3$

$= 4-10+6$

$= 0 \quad \therefore \underset{\sim}{r} \perp \underset{\sim}{q}$.

iii) Find $\begin{pmatrix}l\\m\\n\end{pmatrix}$ such that

$\underset{\sim}{r} \cdot \begin{pmatrix}l\\m\\n\end{pmatrix} = 0$ and $\underset{\sim}{q} \cdot \begin{pmatrix}l\\m\\n\end{pmatrix} = 0$.

$\therefore \begin{pmatrix}4\\-5\\2\end{pmatrix} \cdot \begin{pmatrix}l\\m\\n\end{pmatrix} = 4l-5m+2n = 0$ ①

$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}l\\m\\n\end{pmatrix} = l+2m+3n = 0$

$\therefore 4l+8m+12n = 0$ ②

② − ① $\quad 13m+10n = 0$

$\therefore 13m = -10n$

$m = \dfrac{-10n}{13}$

Q13 cont'd: iii)

$\therefore l + 2\left(\dfrac{-10n}{13}\right) + 3n = 0$

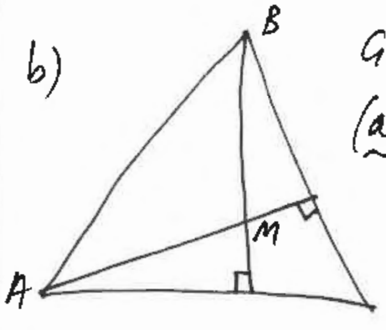
$13l - 20n + 39n = 0$

$13l + 19n = 0$ ③

$l = \dfrac{-19n}{13}$

$\therefore \begin{pmatrix}l\\m\\n\end{pmatrix} = \begin{pmatrix}-\frac{19n}{13}\\ -\frac{10n}{13}\\ n\end{pmatrix}$ or $\begin{pmatrix}-19\\-10\\13\end{pmatrix}$ or $\begin{pmatrix}19\\10\\-13\end{pmatrix}$.

b)



Given $(\underset{\sim}{a}-\underset{\sim}{m}).(\underset{\sim}{b}-\underset{\sim}{c}) = 0$ (perpendicular)

Also $(\underset{\sim}{b}-\underset{\sim}{m}).(\underset{\sim}{a}-\underset{\sim}{c}) = 0$ (given)

RTP: $CM \perp AB$

ie. $(\underset{\sim}{c}-\underset{\sim}{m}).(\underset{\sim}{a}-\underset{\sim}{b}) = 0$

Proof: $(\underset{\sim}{a}-\underset{\sim}{m}).(\underset{\sim}{b}-\underset{\sim}{c}) = 0$ ①

$0 = \cancel{\underset{\sim}{a}.\underset{\sim}{b}} - \underset{\sim}{a}.\underset{\sim}{c} - \underset{\sim}{m}.\underset{\sim}{b} + \cancel{\underset{\sim}{m}.\underset{\sim}{c}}$

and $(\underset{\sim}{b}-\underset{\sim}{m}).(\underset{\sim}{a}-\underset{\sim}{c}) = 0$ ②

$\therefore 0 = \cancel{\underset{\sim}{b}.\underset{\sim}{a}} - \underset{\sim}{b}.\underset{\sim}{c} - \underset{\sim}{m}.\underset{\sim}{a} + \cancel{\underset{\sim}{m}.\underset{\sim}{c}}$

~~Adding~~ ① & ② Subtracting:

$0 = -\underset{\sim}{b}.\underset{\sim}{c} + \underset{\sim}{a}.\underset{\sim}{c} - \underset{\sim}{m}.\underset{\sim}{a} + \underset{\sim}{m}.\underset{\sim}{b}$

$\therefore 0 = \underset{\sim}{c}.(\underset{\sim}{a}-\underset{\sim}{b}) - \underset{\sim}{m}.(\underset{\sim}{a}-\underset{\sim}{b})$

$= (\underset{\sim}{c}-\underset{\sim}{m}).(\underset{\sim}{a}-\underset{\sim}{b})$ ④

$\therefore CM \perp AB$. QED

[Altitudes all meet at M.]

Q13 cont'd:

c) $z = \cos\theta + i\sin\theta$

$z^6+1 = (z^2+1)(z^4-z^2+1)$

$= (z^2+1)(z^4+2z^2+1-3z^2)$

$= (z^2+1)\left((z^2+1)^2-3z^2\right)$

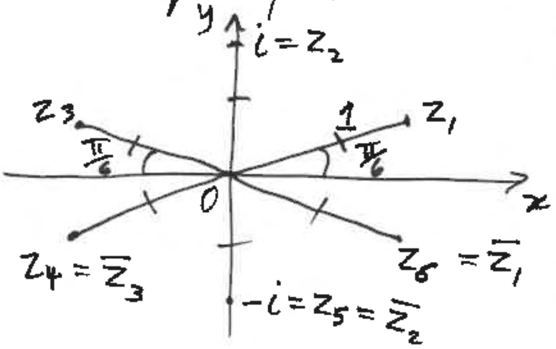
$= (z^2+1)(z^2+1-\sqrt{3}z)(z^2+1+\sqrt{3}z)$

$= (z^2+1)(z^2-\sqrt{3}z+1)(z^2+\sqrt{3}z+1)$

OR let $z^6+1=0 \quad \therefore z^6=-1=\text{cis}\,\pi$

Roots are $z_1=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

then rest equally spaced $\frac{\pi}{3}$.



$\therefore z^6+1=(z-i)(z+i)\left(z-\text{cis}\frac{\pi}{6}\right)\left(z-\text{cis}\left(-\frac{\pi}{6}\right)\right)$

$\times\left(z-\text{cis}\frac{5\pi}{6}\right)\left(z-\text{cis}\left(-\frac{5\pi}{6}\right)\right)$

$=(z^2+1)\left(z^2-2\cos\frac{\pi}{6}z+1\right)\left(z^2-2\cos\frac{5\pi}{6}z+1\right)$

$=(z^2+1)(z^2-\sqrt{3}z+1)(z^2+\sqrt{3}z+1)$

[since $\cos(-\theta)=\cos\theta$] (3)

Q14 a) i) RTP: $a^2+b^2 \geq 2ab$

Proof: Consider the difference:

$a^2+b^2-2ab=(a-b)^2$

≥ 0 since square

$\therefore a^2+b^2 \geq 2ab.$ (1)

ii) RTP: $a^2+b^2+c^2 \geq ab+ac+bc$

Proof: From (i) $a^2+b^2 \geq 2ab$ ①

Similarly, $b^2+c^2 \geq 2bc$ ②

(2) $c^2+a^2 \geq 2ac$ ③

Adding ① + ② + ③

$\cancel{2}a^2+\cancel{2}b^2+\cancel{2}c^2 \geq \cancel{2}ab+\cancel{2}bc+\cancel{2}ac$

$\therefore a^2+b^2+c^2 \geq ab+bc+ca$

iii) RTP: $(a^2+b^2+c^2)^2 \geq 3abc(a+b+c)$

Proof: Consider LHS:

$(a^2+b^2+c^2)(a^2+b^2+c^2)$

$= a^4+b^4+c^4+a^2b^2+a^2c^2+b^2a^2+b^2c^2$
$+ c^2a^2+c^2b^2$

$= a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)$

$\geq a^4+b^4+c^4+2(ab^2c+bc^2a+ca^2b)$

from (ii)

$\geq a^4+b^4+c^4+2(abc(b+c+a))$

$\geq a^2b^2+b^2c^2+c^2a^2+2abc(b+c+a)$

$\geq abc(b+c+a)+2abc(b+c+a)$

$\geq 3abc(a+b+c)$ QED (2)

Q14 cont'd

b) Let $I_n = \int_0^1 x(1-x^3)^n \, dx$.

i) RTP: $I_n = \frac{3n}{3n+2} I_{n-1}$ 3

Proof: $I_n = \int_0^1 x(1-x^3)^n \, dx$

$= \int_0^1 x(1-x^3)(1-x^3)^{n-1} \, dx$

$= \int_0^1 x(1-x^3)^{n-1} \, dx - \int x^4 (1-x^3)^{n-1} \, dx$

$= I_{n-1} + \frac{1}{3} \int_0^1 \underbrace{x^2}_{u} \cdot \underbrace{-3x^2(1-x^3)^{n-1} \, dx}_{dv}$

let $u = x^2$ $\quad dv = -3x^2(1-x^3)^{n-1} \, dx$

$du = 2x \, dx$ $\quad v = \frac{1}{n}(1-x^3)^n$

$= I_{n-1} + \frac{1}{3}\left[uv - \int v \, du\right]$

$= I_{n-1} + \frac{1}{3}\left[\left[x^2 \cdot \frac{(1-x^3)^n}{n}\right]_0^1 - \int_0^1 \frac{(1-x^3)^n}{n} \cdot 2x \, dx\right]$

$I_n = I_{n-1} + \frac{1}{3}\left[\left(\frac{0-0}{n}\right)\right] - \frac{2}{3n} I_n$

$\therefore I_n + \frac{2}{3n} I_n = I_{n-1}$

$\frac{3n+2}{3n} I_n = I_{n-1}$

$\therefore I_n = \frac{3n}{3n+2} I_{n-1}$ QED!

ii) $I_3 = \frac{3(3)}{3(3)+2} \times I_2$

$= \frac{9}{11}\left(\frac{3(2)}{3(2)+2} I_1\right)$

$= \frac{9}{11} \times \frac{6}{8}\left(\frac{3(1)}{3(1)+2}\right) I_0$

14 b) ii) Cont'd

$I_3 = \frac{9}{11} \times \frac{\cancel{6}3}{\cancel{8}4} \times \frac{3}{5} \int_0^1 x(1-x^3)^0 \, dx$

$= \frac{81}{220} \int_0^1 x \, dx$ 2

$= \frac{81}{220}\left[\frac{x^2}{2}\right]_0^1$

$= \frac{81}{220} \times \left(\frac{1}{2} - 0\right)$

$= \frac{81}{440}$

c) i) RTP: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof: Consider let $\alpha = \tan^{-1}x \Rightarrow \tan\alpha = x$

$\beta = \tan^{-1}y \Rightarrow \tan\beta = y$

$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

$\therefore \tan(\tan^{-1}x + \tan^{-1}y) = \frac{x+y}{1-xy}$ 1

$\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

[$\tan^{-1}x$ is a one-to-one fn.]

ii) RTP: let $P(n)$ be the proposition that $\sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{n}{n+1}\right)$.

Proof: See over page!

Proof: RTP:

$\tan^{-1}\left(\frac{1}{2(1)^2}\right)+\tan^{-1}\left(\frac{1}{2(2)^2}\right)+\tan^{-1}\left(\frac{1}{2(3)^2}\right)+\dots+\tan^{-1}\left(\frac{1}{2(n)^2}\right)=\tan^{-1}\left(\frac{n}{n+1}\right).$

Prove P(1) true:

$LHS=\tan^{-1}\left(\frac{1}{2}\right)$ $\quad RHS=\tan^{-1}\left(\frac{1}{1+1}\right)$

$=\tan^{-1}\left(\frac{1}{2}\right)$

$\therefore P(1)$ true. $\quad =LHS$

Assume P(k) true ie.

$\tan^{-1}\left(\frac{1}{2(1)^2}\right)+\tan^{-1}\left(\frac{1}{2(2)^2}\right)+\tan^{-1}\left(\frac{1}{2(3)^2}\right)+\dots+\tan^{-1}\left(\frac{1}{2(k)^2}\right)=\tan^{-1}\left(\frac{k}{k+1}\right)$

for some $k\in\mathbb{N}$.

RTP: $P(k+1)$ true: ie.

$\tan^{-1}\left(\frac{1}{2(1)^2}\right)+\dots+\tan^{-1}\left(\frac{1}{2(k)^2}\right)+\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$

$=\tan^{-1}\left(\frac{k+1}{k+2}\right).$

Proof: Consider the LHS of $P(k+1)$

$\therefore \underbrace{\tan^{-1}\left(\frac{1}{2(1)^2}\right)+\dots+\tan^{-1}\left(\frac{1}{2(k)^2}\right)}+\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$

$=\tan^{-1}\left(\frac{k}{k+1}\right)+\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$ using $P(k)$

$=\tan^{-1}\left(\dfrac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\cdot\frac{1}{2(k+1)^2}\right)}\right)$ $\quad |x|<1$, $|y|<1$

$=\tan^{-1}\left(\dfrac{\frac{2(k+1)k+1}{\cancel{2}(k+1)^2}}{\frac{2(k+1)^3-k}{\cancel{2}(k+1)^{\cancel{3}\,1}}}\right)$

Q14 cont'd

c) ii) cont'd:

$=\tan^{-1}\left(\frac{(2k^2+2k+1)(k+1)}{2(k+1)^3-k}\right)$

$=\tan^{-1}\left(\frac{(2k^2+2k+1)(k+1)}{2(k^3+3k^2+3k+1)-k}\right)$

$=\tan^{-1}\left(\frac{(2k^2+2k+1)(k+1)}{2k^3+6k^2+6k+2-k}\right)$

$=\tan^{-1}\left(\frac{(2k^2+2k+1)(k+1)}{2k^3+6k^2+5k+2}\right)$

$=\tan^{-1}\left(\frac{\cancel{(2k^2+2k+1)}(k+1)}{\cancel{(2k^2+2k+1)}(k+2)}\right)$ check this step!

$=\tan^{-1}\left(\frac{k+1}{k+2}\right)$

$=RHS$ of $P(k+1)$ 3

$\therefore P(k+1)$ is true

$\therefore P(n)$ true by Math Induction.

iii) $\lim_{n\to\infty}\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right)$

$=\lim_{n\to\infty}\tan^{-1}\left(\frac{n}{n+1}\right)$

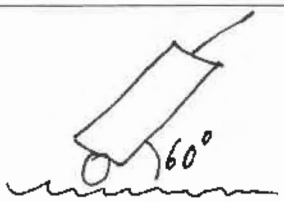
$\approx\lim_{n\to\infty}\tan^{-1}\left(\frac{n}{n}\right)$

$\approx\lim_{n\to\infty}\tan^{-1}1$

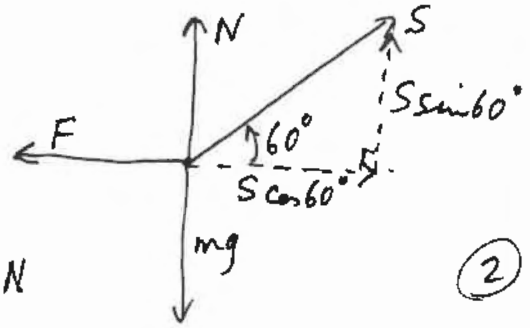
$=\frac{\pi}{4}.$ 1

Q 15

a)



i)



$F = \mu N$ (2)

Horizontally: $S\cos 60° - F = 0$

$F = S\cos 60° = \frac{1}{2}S.$

Vertically: $N + S\sin 60° - mg = 0$

$N = mg - \frac{\sqrt{3}}{2}S.$

ii) $\mu = ?$ $S, m, g.$ $\mu = \frac{F}{N}$

$$\mu = \frac{\frac{1}{2}S}{mg - \frac{\sqrt{3}}{2}S}$$ (2)

$$\therefore \mu = \frac{S}{2mg - \sqrt{3}S}$$

b) $v^2 = x(6-x) = 6x - x^2$

i) RTP: form is $\ddot{x} = -n^2(x-c)$ (2)

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}(6x - x^2)\right)$

$= 3 - x$

$= -1^2(x-3)$ which is

SHM $c = 3,\ n = 1$

15 cont'd

b) ii) Centre is $x = 3.$ (2)

End points when $v = 0$

$0 = 6x - x^2 = x(6-x)$

$x = 0$ or $x = 6$, centre 3

$\therefore$ Amplitude is 3 units.

iii) Period $= \frac{2\pi}{n} = \frac{2\pi}{1} = 2\pi.$ (1)

iv) Max speed occurs at centre

when $x = 3$ $|v| = \sqrt{3(6-3)}$

$= 3.$ (1)

v) $t = 0,\ v = 3,\ x = 3.$

$\therefore \quad v = +\sqrt{x(6-x)}$

$\therefore \quad \frac{dx}{dt} = \sqrt{6x - x^2}$

$\therefore \displaystyle\int_3^x \frac{1}{\sqrt{6x - x^2}}\,dx = \int_0^t dt$

$\displaystyle\int_3^x \frac{1}{\sqrt{9 - (x-3)^2}}\,dx = t$ (2)

$\left[\sin^{-1}\left(\frac{x-3}{3}\right)\right]_3^x = t$

$\sin^{-1}\frac{x-3}{3} - \sin^{-1}0 = t$

$\therefore \frac{x-3}{3} = \sin t$

$x = 3\sin t + 3$.

vi) $t = 3,\ x = 3\sin 3 + 3 \doteqdot 0.423360\ldots$ to Right

(3) $v = 3\cos 3 = -2.9699\ldots$ to left

$\ddot{x} = -3\sin 3 = -0.4233\ldots$

$\therefore$ P is ≈ 0.423 to right of 0, speeding up, going left.

Q16 a) $u = \cos 2\theta$

$du = -2\sin 2\theta \, d\theta$

$u = 1, \quad \theta = \pi \text{ or } 0$

$u = \frac{1}{2} \quad \theta = \frac{\pi}{6}$

$$\therefore \int_{\frac{1}{2}}^{1} \sqrt{\frac{1-u}{1+u}} \, du$$

$$= \int_{\frac{\pi}{6}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \times -2\sin 2\theta \, d\theta$$

$$= \int_{0}^{\frac{\pi}{6}} \sqrt{\frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}} \times 2\sin\theta\cos\theta \, d\theta$$

$$= \int_{0}^{\frac{\pi}{6}} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= \int_{0}^{\frac{\pi}{6}} \frac{\sin\theta}{\cancel{\cos\theta}} \cdot 2\sin\theta \cancel{\cos\theta} \, d\theta$$

$$= \int_{0}^{\frac{\pi}{6}} 2\sin^2\theta \, d\theta$$

(4)

$$= \int_{0}^{\frac{\pi}{6}} 1 - \cos 2\theta \, d\theta$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3} - \left(0 - \frac{1}{2}\sin 0\right)$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}.$$

b) $y = \frac{\tan x}{x}, \quad y = \frac{1}{x}$

i) Sub $\left(\frac{\pi}{4}, \frac{4}{\pi}\right)$: $y = \frac{4}{\pi}$, RHS $= \frac{\tan\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{4}{\pi}$ ✓

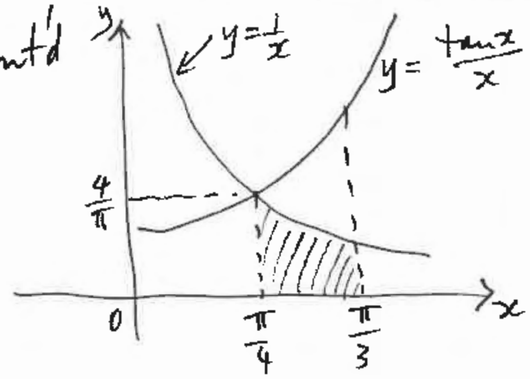
$y = \frac{1}{\frac{\pi}{4}} = \frac{4}{\pi}$ ✓ (1)

$\therefore$ RHS = LHS for both

$\therefore$ Intersects there.

b) cont'd

ii)



Clearly from diagram,

Area below $y = \frac{\tan x}{x}$ > Area below $y = \frac{1}{x}$

between $x = \frac{\pi}{4}$ and $\frac{\pi}{3}$.

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x} \, dx > \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{x} \, dx$$

$$> \left[\ln x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

(3)

$$> \ln\frac{\pi}{3} - \ln\frac{\pi}{4}$$

$$> \ln\left(\frac{\frac{\pi}{3}}{\frac{\pi}{4}}\right)$$

$$> \ln\frac{4}{3}. \quad \text{QED!}$$

c) $\ddot{x} = \frac{v}{50}(50 - v)$

$t = 0, \quad x = 0, \quad v = 100$

i) let $\cancel{v}\frac{dv}{dx} = \frac{\cancel{v}}{50}(50 - v)$

$$\int_{100}^{v} \frac{dv}{50 - v} = \frac{1}{50}\int_0^x dx$$

$$\left[-\ln|50 - v|\right]_{100}^{v} = \frac{1}{50}x$$

P.T.O.

Q16 cont'd

c) i) cont'd

$\therefore -\left[\ln|50-v| - \ln|50-100|\right] = \frac{1}{50}x$

$\therefore -\left[\ln\left|\frac{50-v}{-50}\right|\right] = \frac{1}{50}x$

$\ln\left|\frac{-50}{50-\cancel{x}}\right| = \frac{1}{50}x$ (3)

$\frac{-50}{50-v} = e^{\frac{1}{50}x}$

$-50e^{-\frac{1}{50}x} = 50-v$

$\therefore v = 50 + 50e^{-\frac{1}{50}x}$

$\therefore v = 50\left(1+e^{-\frac{1}{50}x}\right)$

ii) $\frac{dx}{dt} = 50\left(1+e^{-\frac{1}{50}x}\right)$

$\int_0^x \frac{dx}{1+e^{-\frac{1}{50}x}} = \int_0^t 50\,dt$ (2)

$\int_0^x \frac{e^{\frac{1}{50}x}}{e^{\frac{1}{50}x}+1}\,dx = 50t$

$\left[\ln\left|e^{\frac{1}{50}x}+1\right|\right]_0^x = 50t$

$\ln\left|e^{\frac{1}{50}x}+1\right| - \ln\left|e^0+1\right| = 50t$

$\ln\left|\frac{e^{\frac{1}{50}x}+1}{2}\right| = 50t$

$\frac{e^{\frac{1}{50}x}+1}{2} = e^{50t}$

$e^{\frac{1}{50}x}+1 = 2e^{50t}$

16 c) ii) cont'd

$e^{\frac{1}{50}x} = 2e^{50t}-1$

$\therefore \frac{1}{50}x = \ln\left|2e^{50t}-1\right|$

$x = 50\ln\left|2e^{50t}-1\right|$

iii) $x = 25,\ t = ?$

$25 = 50\ln\left|2e^{50t}-1\right|$

$\frac{1}{2} = \ln\left|2e^{50t}-1\right|$

$e^{\frac{1}{2}} = 2e^{50t}-1$ (2)

$\therefore e^{50t} = \frac{e^{\frac{1}{2}}+1}{2}$

$\therefore 50t = \ln\left|\frac{\sqrt{e}+1}{2}\right|$

$t = \frac{1}{50}\ln\left|\frac{\sqrt{e}+1}{2}\right|$

$\approx 0.005618596\ldots$ units!

Student Number ....

ASCHAM SCHOOL

YEAR 12 Trial Mathematics Extension 2 Exam

MULTIPLE-CHOICE ANSWER SHEET

1.	A ●	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ●
3.	A ●	B ○	C ○	D ~~●~~
4.	A ○	B ○	C ○	D ●
5.	A ○	B ○	C ●	D ○
6.	A ○	B ○	C ○	D ●
7.	A ○	B ○	C ○	D ●
8.	A ○	B ●	C ○	D ○
9.	A ○	B ○	C ●	D ○
10.	A ○	B ●	C ○	D ○